DEVELOPMENT OF HYBRID ANALYTICAL—NUMERICAL METHODS OF CALCULATING HEAT AND MASS TRANSFER

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The expediency of using approximate numerical or analytical methods of calculation with respect to elliptical coordinates and accurate methods with respect to unidirectional parabolic variables is demonstrated.

Much of the scientific research of A. V. Lykov was devoted to the development of an analytical theory of heat conduction. As well as the broad use of classical methods of solving mathematical-physics boundary problems, Lykov developed a theory of application of integral transformations to the solution of problems of noncoupled and coupled heat and mass transfer.

The modern trend to complication of mathematical models of thermophysical processes, associated with a more profound formulation of well-known or new phenomena of linear and nonlinear transport, entails the development of improved methods of calculation for the given problems. The development of such methods is unimaginable without profound study and reinterpretation of Lykov's scientific legacy for the current level of development of thermophysics.

In studying heat- and mass-transfer processes, the temperature is the basic physical quantity. In the general case, it is not only a function of the input thermophysical parameters, but also depends on the coordinates of the current point M(x, y, z) and the time t, i.e., is a function of four arguments. In the systematic application of accurate or approximate (numerical or analytical) applied-mathematics apparatus with respect to these variables, the classification of these variables into parabolic, elliptical, and hyperbolic types proposed in [1] proves useful; this terminology is adopted by analogy with the classification of partial differential equations. For example, in the heat-transfer equations with liquid flow in straight tubes with constant cross section D (y, z ∈ D)

$$c\gamma\left(\frac{\partial T}{\partial t}+w(y,\ z)\frac{\partial T}{\partial x}\right)=\operatorname{div}(\lambda\operatorname{qrad}T)+q(M,\ t), \tag{1}$$

$$0\leqslant t<\infty,\ 0\leqslant x<\infty,\ \operatorname{div}(\lambda\operatorname{qrad}T)=\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right),$$

the variables t and x are unidirectional parabolic coordinates, while y and z are bidirectional elliptical coordinates varying in the finite region D.

It is expedient to introduce the property of integral transformation

$$\bar{f}(\alpha)=\int_a^b f(x)\,k(\alpha,\ x)\,dx \tag{2}$$

where k(α, x) is the kernel of the transformation, which allows some information to be obtained regarding the original f(x), i.e., regarding the temperature field T(x, y, z, t) from its mapping. This possibility of using integral transformations to investigate heat-transfer problems was first noted by Lykov, whose wrote [2] that "the transition from differential equations to algebraic equations consists not only in replacing the partial derivative $\partial^m T/\partial x^n$ by the expression $p^m\bar{T}$ but also in introducing additional relations which take account of the boundary conditions, i.e., the interaction of the body with the surrounding medium. Physically, this means transition from the actual values of the quantities being

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investigated (differential equations and uniqueness conditions) to mean values taken in accordance with the conditions of the specific physical problem by methods of operator transformation."

Suppose that the kernel $k(\alpha, x)$ is nonorthogonal with unity, i.e., $\int_a^b k(\alpha, x)\cdot 1dx \neq 0$; then

$$\frac{\int_a^b f(x)\, k(\alpha, x)\, dx}{\int_a^b k(\alpha, x)\, dx} = \langle \bar{f}(\alpha) \rangle \tag{3}$$

is the integral mean of the desired physical quantity $f(x)$ in the interval $[a; b]$ with averaging kernel $k(\alpha, x)$. When $k(\alpha, x) \equiv 1$, ordinary integral averaging is obtained.

Setting $w(y, z) = k$, the kernel of transformation, for the temperature field in Eq. (1), it is found that

$$\langle T(x, t) \rangle = \frac{\iint_D T(x, y, z, t)\, w(y, z)\, dydz}{\iint_D w(y, z)\, dydz} \tag{4}$$

is the mean mass temperature. Below, replacing $T(x, y, z, t)$ by $\langle T(x, t)\rangle$, a simplified model of the solution of a generalized problem of Graetz–Nusselt type is obtained; now, setting $k(\alpha, x) = k(p, t) = \exp(-pt)$, the expression

$$\bar{T}_\tau(x, y, z, p) = \frac{\int_0^\tau T(x, y, z, t)\, k(p, t)\, dt}{\int_0^\tau k(p, t)\, dt} \tag{5}$$

is the integral mean temperature in the time interval $\tau > 0$ with averaging kernel $k(p, t) = \exp(-pt)$. In the limit as $\tau \to \infty$, it is found that

$$\lim_{\tau\to\infty} \bar{T}_\tau(x, y, z, p) = \frac{\int_0^\infty T(x, y, z, t) \exp(-pt)\, dt}{\lim\limits_{\tau\to\infty} (1 - \exp(-p\tau))\, \dfrac{1}{p}} = p\bar{T}(x, y, z, p) = \bar{T}^*(x, y, z, p), \tag{6}$$

where $\bar{T}(x, y, z, p)$ is the Laplace transform and $\bar{T}^*(x, y, z, p)$ is the Carson–Laplace transform. An interesting direct relation is obtained between $\bar{T}^*(x, y, z, p)$ and the mean temperature over the time with the exponentially decreasing kernel $\exp(-pt)$, $p > 0$, over the whole time interval of the nonsteady process. Hence, if the Laplace transformation does not include a physical interpretation, multiplication of this expression by p leads to the specific content of the quantity $\bar{T}^*(x, y, z, p)$. This relation gives real physical meaning to the well-known unique correspondence at the two limiting points

$$\lim_{p\to 0} \bar{T}^*(x, y, z, p) = \lim_{p\to 0} p\bar{T}(x, y, z, p) = \lim_{t\to\infty} T(x, y, z, t), \tag{7}$$

$$\lim_{p\to\infty} \bar{T}^*(x, y, z, p) = \lim_{p\to\infty} p\,\bar{T}(x, y, z, p) = \lim_{t\to 0} T(x, y, z, t). \tag{8}$$

On the basis of the property in Eq. (8), by introducing an oscillatory component of the transform $\bar{T}^*$ in the vicinity of an infinitely remote point p, Lykov obtained, for the first time, expressions for calculating the temperature at small Fo permitting, for example, the calculation of the temperature when Fo = 40 by means of two or three terms, rather than the 40 terms in the accurate solutions [2].

The application of integral transformations to a differential operator of second order with respect to the elliptical coordinates in the nonsteady heat-conduction equation leads, for the mapping (transform), to solution of the Cauchy problem for a first-order equation. The original (prototype), i.e., the solution of the initial problem in the case of integral transformation in a finite interval, is found in the form of a functional series, where summation is taken over a discrete spectrum of eigenvalues and eigenfunctions. For semi-infinite and infinite intervals — $a = 0$, $b = \infty$ or $a = -\infty$, $b = \infty$ in Eq. (2) — transformation from the mapping to the original is by means of summation over a continuous spectrum, i.e., the solution of the problem is expressed in terms of an integral.

In the first case, the application of accurate applied-mathematics apparatus leads to representation of the set of solutions in the form of elements of functional space, the bases of which are eigenfunctions of the differential thermal-sensing operator with respect to a bidirectional elliptical coordinate; their linear independence and orthogonality is determined by the eigenvalue spectrum. These eigenvalues are found by solving the so-called characteristic equations, which correspond to a specific type of boundary conditions and the geometry of the body. For example, the bases of the functional spaces in which the temperature fields inside a plate, a sphere, and a spherical shell are found are various systems of trigonometric sine or cosine functions, while for solid and hollow cylinders they are systems of Bessel functions. On the other hand, it is known from mathematical analysis that these functions are accurately expressed by converging power series, i.e., the power polynomials may be approximated with any accuracy. Hence, the representation of the solution in such a functional space, the coordinate bases of which are power polynomials, will be well founded and goal-directed.

Such methods of calculation of the mathematical models of nonsteady heat-transfer processes include the method of combined application of Laplace–Carson integral transformations with respect to parabolic variables and the finite-difference method with realization of orthogonal projection of the discrepancy over the whole region of variation of the bidirectional elliptical coordinates. The advantage of this approach is the more reliable and unchallengeable calculation of heat-conduction problems in multidimensional bodies of nonclassical form, for which it is not possible to write the characteristic equations in explicit form and find the eigenfunctions. Generalized problems of Graetz–Nusselt type may be effectively investigated by this method, and the given problems may also be solved in those cases where the differential operator $\operatorname{div}(\lambda \operatorname{grad} T)$ with respect to the bidirectional elliptical coordinates λ includes a function of these same variables, i.e., it is possible to find the temperature fields in tubes with turbulent flow, in inhomogeneous solids, and in materials of composite structure [2-4].

The temperature field in a plate (m = 0), a cylinder (m = 1), and a sphere (m = 2) is now determined as a solution of the problem

$$\frac{\partial T}{\partial \mathrm{Fo}} = \frac{1}{\xi^m}\frac{\partial}{\partial \xi}\left(\xi^m \frac{dT}{\partial \xi}\right) + \frac{q_v(\xi,\ \mathrm{Fo})R^2}{\lambda}, \quad [T(\xi,\ \mathrm{Fo})]_{\mathrm{Fo}=0} = T_0, \tag{9}$$

$$\left\{\frac{\partial T}{\partial \xi} + \mathrm{Bi}\, T(\xi,\ \mathrm{Fo})\right\}_{\xi=1} = \mathrm{Bi}\left[\varphi(\mathrm{Fo}) + \frac{q_c(\mathrm{Fo})}{\alpha}\right], \quad \left(\frac{\partial T}{\partial \xi}\right)_{\xi=0} = 0, \tag{10}$$

where $0 \le \xi = r/R \le 1$ for m = 1; 2; $-1 \le \xi = x/R \le 1$, $\mathrm{Fo} = at/R^2$, $\mathrm{Bi} = \alpha R/\lambda$. In the space of Laplacian transforms

$$\frac{d}{d\xi}\left(\xi^m \frac{d\bar{T}}{d\xi}\right) - [p\bar{T}(\xi,\ p) - T_0]\,\xi^m + \frac{\bar{q}_v(\xi,\ p)R^2\xi^m}{\lambda} = 0, \tag{11}$$

$$\left\{\frac{d\bar{T}}{d\xi} + \mathrm{Bi}\bar{T}(\xi,\ p)\right\}_{\xi=1} = \mathrm{Bi}\left[\bar{\varphi}(p) + \frac{\bar{q}_c(p)}{\alpha}\right], \quad \left(\frac{d\bar{T}}{d\xi}\right)_{\xi=0} = 0. \tag{12}$$

Determining the accurate solution of the boundary problem in Eqs. (11) and (12) with respect to the elliptical variable ξ and transforming to the region of originals with respect to the time at each fixed m, as was done by Lykov in his investigations, the solution of the initial problem may be found. In this approach, the basic difficulty is associated with inverse transformation to the original; fairly complex mathematical maneuvers are required here. These special mathematical investigations by Lykov made a scientific contribution to the classical theory of operational calculus.

The solution of Eqs. (11) and (12) is now continued by the method of orthogonal projection of the discrepancy. In the class of stabilizing input functions of the thermal load

$$\lim_{p\to 0} p\left[\bar{\varphi}(p)+\frac{\bar{q}_c(p)}{\alpha}\right]=\lim_{\mathrm{Fo}\to\infty}\left[\varphi(\mathrm{Fo})+\frac{q_c(\mathrm{Fo})}{\alpha}\right]= \quad (13)$$

$$=T_c+\frac{q_c}{\alpha}=\mathrm{const},$$

$$\lim_{p\to 0} p\,\bar{q}_v(\xi,\ p)=\lim_{\mathrm{Fo}\to\infty} q_v(\xi,\ \mathrm{Fo})=q_v(\xi), \quad (14)$$

it is evident that

$$\lim_{p\to 0} p\,\bar{T}(\xi,\ p)=\lim_{\mathrm{Fo}\to\infty} T(\xi,\ \mathrm{Fo})=T^*(\xi). \quad (15)$$

Multiplying Eqs. (11) and (12) by p and passing to the limit as $p \to 0$, it follows that

$$\frac{d}{d\xi}\left(\xi^m\frac{dT^*}{d\xi}\right)+\frac{q_v(\xi)R^2\xi^m}{\lambda}=0, \quad (16)$$

$$\left\{\frac{dT^*}{d\xi}+\mathrm{Bi}\,T^*(\xi)\right\}_{\xi=1}=\mathrm{Bi}\left[T_c+\frac{q_c}{\alpha}\right],\quad\left(\frac{dT^*}{d\xi}\right)_{\xi=0}=0, \quad (17)$$

and hence it follows that when $q_V(\xi) = q_V = \mathrm{const}$, $q_V(\xi) = q_V(1 - \xi^2)$,

$$T^*(\xi)=T_c+\frac{q_c}{\alpha}+\frac{q_v R^2}{2\lambda(m+1)}\left(\frac{\mathrm{Bi}+2}{\mathrm{Bi}}-\xi^2\right), \quad (18)$$

$$T^*(\xi)=T_c+\frac{q_c}{\alpha}+\frac{q_v R^2}{4\lambda(m+1)(m+3)}\left\{\frac{\mathrm{Bi}(m+5)+8}{\mathrm{Bi}}-\right.$$

$$\left.-2\xi^2(m+3)+(m+1)\xi^4\right\}. \quad (19)$$

By the method of choosing the optimal system of basis coordinates [5] with two forms of the steady distribution of the internal heat sources following the transient conditions, the temperature field in the heat-liberating elements (HLE) is found as the solution of Eq. (11) accurately satisfying the boundary conditions in Eq. (12), in the form

$$\bar{T}_n(\xi,\ p)=\bar{\varphi}(p)+\frac{\bar{q}_c(p)}{\alpha}+\bar{a}_1(p,\ m)\left(\frac{\mathrm{Bi}+2}{\mathrm{Bi}}-\xi^2\right)+$$

$$+\sum_{k=2}^{n}\bar{a}_k(p,\ m)(1-\xi^2)^2\xi^{2(k-2)}, \quad (20)$$

$$\bar{T}_n(\xi,\ p)=\varphi(p)+\frac{\bar{q}_c(p)}{\alpha}+\bar{a}_1(p,\ m)\left[\frac{\mathrm{Bi}(m+5)+8}{\mathrm{Bi}}-2\xi^2(m+3)+\right.$$

$$\left.+\xi^4(m+1)\right]+\sum_{k=2}^{n}\bar{a}_k(p,\ m)(1-\xi^2)^2\xi^{2k}. \quad (21)$$

The procedure for determining the mapping coefficients $\bar{a}_k(p, m)$ ($k = 1, 2, \ldots, n$) and their physical interpretation was outlined in [3]. For the case when $q_V(\ell, \mathrm{Fo}) = q_V = \mathrm{const}$, it follows from the abbreviated first-order system that

$$a_1(p,\ m)=\frac{\left\{T_0-p\left[\bar{\varphi}(p)+\frac{\bar{q}_c(p)}{\alpha}\right]\right\}}{2(m+1)[p+A(\mathrm{Bi},\ m)]}+\frac{q_v R^2}{2\lambda(m+1)}\left\{\frac{1}{p}-\frac{1}{p+A(\mathrm{Bi},\ m)}\right\}, \quad (22)$$

where the expression

$$A(\mathrm{Bi},\ m)=\frac{\mathrm{Bi}(m+1)(m+5)+(\mathrm{Bi}+m+3)}{2\,\mathrm{Bi}^2+2\,\mathrm{Bi}(m+5)+(m^2+8m+15)} \quad (23)$$

gives good agreement with the square of the first root (μ_1^2) of the three characteristic equations [3].

With constant boundary conditions

$$\left(\varphi(\mathrm{Fo}) + \frac{q_c(\mathrm{Fo})}{\alpha} = T_c + \frac{q_c}{\alpha}\right)$$

the relative excess temperature is

$$\theta(\xi,\ \mathrm{Fo},\ m) = \frac{T(\xi,\ \mathrm{Fo}) - T_0}{T_c + \frac{q_c}{\alpha} - T_0} = 1 + \frac{A(\mathrm{Bi},\ m)}{2(m+1)}\left(\frac{\mathrm{Bi}+2}{\mathrm{Bi}} - \right.$$

$$\left. - \xi^2\right)\exp[-A(\mathrm{Bi},\ m)\,\mathrm{Fo}] + \frac{\mathrm{Po}}{2(m+1)}\{1 - \exp[-A(\mathrm{Bi},\ m)\,\mathrm{Fo}]\} \times \tag{24}$$

$$\times\left(\frac{\mathrm{Bi}+2}{\mathrm{Bi}} - \xi^2\right),\quad \mathrm{Po} = \frac{q_v R^2}{\lambda\left(T_c + \frac{q_c}{\alpha} - T_0\right)}.$$

The expression in Eq. (24) practically coincides with the precise formula for Fo ≥ 0.1 and tends to the precise asymptotic solution in Eq. (18) as Fo → ∞.

If $\bar{a}_k(p,\ m)$ is calculated from the complete determining system, and then the limit as p → 0 is found, the result obtained is

$$\lim_{p\to 0} p\,\bar{a}_1(p,\ m) = \lim_{\mathrm{Fo}\to\infty} a_1(\mathrm{Fo},\ m) = \frac{q_v R^2}{2\lambda(m+1)}$$

in the first case and

$$\lim_{p\to 0} p\,\bar{a}_1(p,\ m) = \lim_{\mathrm{Fo}\to\infty} a_1(\mathrm{Fo},\ m) = \frac{q_v R^2}{4\lambda(m+1)(m+3)}$$

in the second. For k ≥ 2, in both cases

$$\lim_{p\to 0} p\,\bar{a}_k(p,\ m) = \lim_{\mathrm{Fo}\to 0} a_k(\mathrm{Fo},\ m) = 0,$$

i.e., the approximate solutoins in Eqs. (20) and (21) in the region of the originals tends to the accurate solution as Fo → ∞. Calculation in the second, third, and subsequent approximations only refines the temperature variation in the period of transient conditions (0 ≤ Fo ≤ 0.1). Note that, in determining the temperature at points of the functional spaces (for example for a cylindrical HLE from a system of Bessel functions), such effective solutions with the above properties cannot be obtained.

Setting

$$\mathrm{div}(\lambda\,\mathrm{grad}\,T) = \frac{\lambda}{r^m}\frac{\partial}{\partial r}\left(r^m\frac{\partial T}{\partial r}\right),\quad q_v = 0,\ m = 0;\ 1,$$

in Eq. (1), and applying a Laplace–Carson transformation with respect to the variables Fo, X = (1/Pe)(x/R)

$$\bar{T}^*(\xi,\ s,\ p) = sp\int_0^\infty\int_0^\infty T(\xi,\ X,\ \mathrm{Fo})\exp[-(sX + p\,\mathrm{Fo})]\,dX\,d\mathrm{Fo}$$

it is found that

$$\frac{d}{d\xi}\left(\xi^m\frac{d\bar{T}^*}{d\xi}\right) - [p + sw(\xi)]\bar{T}^*(\xi,\ s,\ p)\,\xi^m + [p\,T_0 + sw(\xi)\bar{\varphi}_0(p)]\,\xi^m = 0, \tag{25}$$

where $w(\xi) = 2(1 - \xi^2)$ for laminar Poiseuille flow with m = 1 and $w(\xi) = 6\xi(1 - \xi)$ for a plane channel (m = 0) of width R. For a plane liquid layer running off an inclined plane, $w(\xi) = 3/2\xi(2 - \xi)$.

Suppose that the tube wall is thermally thin, and the heat transfer of the liquid flux with the external medium is described by generalized boundary conditions of the third kind

$$\left\{\frac{d\bar{T}^*}{d\xi}+\mathrm{Bi}\bar{T}^*(\xi,\ s,\ p)\right\}_{\xi=1}=\mathrm{Bi}\left[\bar{\varphi}^*(s,\ p)+\frac{\bar{q}_c^*(s,\ p)}{\alpha}\right],$$
$$\left(\frac{d\bar{T}^*}{d\xi}\right)_{\xi=0}=0. \tag{26}$$

For a plane channel and a liquid layer running off an inclined plane, the condition $(d\bar{T}^*/d\xi)_{\xi=0} = 0$ means that the lower wall is adiabatic. With any profile $w(\xi)$, the temperature field in the liquid flux accurately satisfying the boundary conditions in Eq. (26) is found in the form

$$\bar{T}_n^*(\xi,\ s,\ p)=\bar{\varphi}^*(s,\ p)+\frac{\bar{q}_c^*(s,\ p)}{\alpha}+\bar{a}_1^*(s,\ p)\left(\frac{\mathrm{Bi}+2}{\mathrm{Bi}}-\xi^2\right)+$$
$$+\sum_{k=2}^{n}\bar{a}_k^*(s,\ p)(1-\xi^2)^2\xi^{2(k-2)}. \tag{27}$$

After determining $\bar{a}_k^*(s, p)$ by the method of orthogonal projection of the discrepancy and transition to the region of originals, the temperature field exponentially stabilizing over time and along the liquid flow is found. More details on this approach may be found in [3]; below, a simplified method of investigation is outlined. Let

$$\langle\bar{T}^*(s,\ p)\rangle=\frac{(m+1)\int_0^1\bar{T}^*(\xi,\ s,\ p)\,w(\xi)\,\xi^m\,d\xi}{(m+1)\int_0^1 w(\xi)\,\xi^m\,d\xi},$$

$$(m+1)\int_0^1 w(\xi)\,\xi^m\,d\xi=1$$

and also

$$\langle\bar{T}^*(s,\ p)\rangle=\bar{T}^*(s,\ p)\doteq T(X,\ \mathrm{Fo}),$$

where $T(X, \mathrm{Fo})$ is the mean mass temperature with velocity profile $w(\xi)$. As a simplification of the mathematical model, it is assumed that

$$(m+1)\int_0^1 T(\xi,\ X,\ \mathrm{Fo})\,\xi^m\,d\xi=T(X,\ \mathrm{Fo}),\ T(X,\ \mathrm{Fo})=[T(\xi,\ X,\ \mathrm{Fo})]_{\xi=1}.$$

Under these assumptions, after multiplying Eq. (25) by (m + 1) and integrating with respect to ξ from 0 to 1, taking account of the boundary condition in Eq. (26), an algebraic equation in $\bar{T}^*(s, p)$ is obtained; hence,

$$\bar{T}^*(s,\ p)=\frac{pT_0+s\,\bar{\varphi}_0(p)}{p+s+(m+1)\,\mathrm{Bi}}+$$
$$+\frac{(m+1)\,\mathrm{Bi}\left[\bar{\varphi}^*(s,\ p)+\frac{\bar{q}_c^*(s,\ p)}{\alpha}\right]}{p+s+(m+1)\,\mathrm{Bi}}. \tag{28}$$

The temperature variation over time and along the liquid flow when $\varphi_0(\mathrm{Fo}) = T_0$, $\varphi(X, \mathrm{Fo}) = T_c$, $q_c(X, \mathrm{Fo}) = q_c$ is found from the formula

$$T(X,\ \mathrm{Fo},\ m)=T_0+\left(T_c+\frac{q_c}{\alpha}-T_0\right)\{1-\exp[-(m+1)\,\mathrm{Bi}\,f]\}, \tag{29}$$

where $f = \mathrm{Fo}$ for $X > \mathrm{Fo}$ and $f = X$ for $X < \mathrm{Fo}$. Graphs of the relative temperature

$$\Theta(X,\ \mathrm{Fo})=\frac{T(X,\ \mathrm{Fo})-T_0}{T_c+q_c/\alpha-T_0}$$

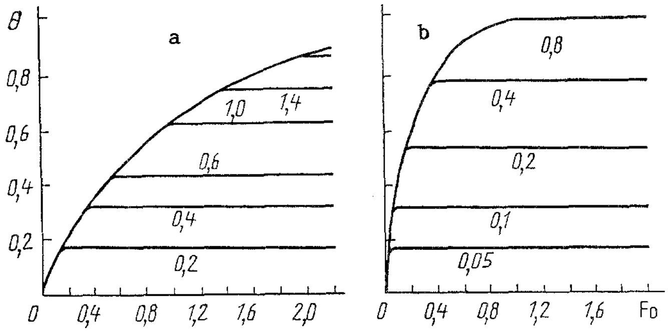

Fig. 1. Variation in relative excess temperature over time in fixed cross sections X (given by the numbers on the curves): a) Bi = 0.5; b) 2.

for m = 1, Bi = 0.5, and Bi = 2 are shown in Fig. 1. Equation (29) coincides satisfactorily with the calculation of the temperature by the unsimplified model for small Bi ($0 < Bi \leq 1$). Using Eq. (28), the temperature with other forms of other thermal load may be calculated from the simplified model, when the temperature variation along the current radius or along the channel height is not taken into account, as is often done in solar technology [6].

Setting $q_V = 0$, $m = 0$, $0 \leq \xi = x < 1$, $\tau = Fo$ in Eq. (9), the heat-conduction equation obtained is approximated by a finite-difference formula

$$\frac{T_k^{i+1} - T_k^i}{\Delta\tau} = \frac{T_{k+1}^i - 2T_k^i + T_{k-1}^i}{(\Delta x)^2}, \tag{30}$$

where $T_k^i = T(\Delta x \cdot k, \Delta\tau \cdot i)$, $\Delta x = 1/n$, T_0^i, T_n^i, and T_k^0 are values of the known functions $\varphi_1(\tau)$, $\varphi_2(\tau)$, $f(x) = T(x, 0)$ with boundary conditions of the first kind.

In [6], a numerical experiment to solve Eq. (30) when $\varphi_1(\tau) = \varphi_2(\tau) = 0$ and with the initial temperature profile in the form of the lateral sides of an isosceles triangle was outlined. For n = 20, $\Delta x = 1/n = 0.05$, with $\Delta\tau/(\Delta x)^2 = 5/11$ good agreement with the accurate data was obtained; when $\Delta\tau/(\Delta x)^2 = 5/9$, there was rapid accumulation of the calculational error, which reached an unacceptable level at the middle of the plate (up to 150%). Although the numbers 5/11 and 5/9 are close, opposite results are obtained, because the stability condition for Eq. (30) is

$$\beta = \frac{\Delta\tau}{(\Delta x)^2} \leqslant \frac{1}{2}. \tag{31}$$

Decrease in Δx with the aim of improving the convergence of the numerical solution may have the opposite result. Evidently, with $\Delta\tau$ as small as may be desired ($\Delta\tau \to 0$), the limiting analog of Eq. (30) will be stable with any Δx, i.e., it is more expedient to solve the system of differential equations

$$\frac{dT_k}{d\tau} = \frac{T_{k+1} - 2T_k + T_{k-1}}{(\Delta x)^2}, \quad k = 1, 2, \ldots, (n-1). \tag{32}$$

Solution of Eq. (32) in matrix form when $\varphi_1(\tau) = \varphi_2(\tau) = T_c$, $f(x) = T_0$ gives the first (n = 2), second (n = 4), and third (n = 6) approximations at the middle of the plate (x = 0.5) in the form

$$\Theta^{(1)}(\tau) = \frac{T(\tau) - T_0}{T_c - T_0} = \exp(-8\tau),\ \Theta^{(2)}(\tau) = 1.207\exp(-9.333\tau) - 0.207\exp(-54.627\tau),$$
$$\Theta^{(3)}(\tau) = 1.244\exp(-9.648\tau) - 0.333\exp(-72\tau) + 0.089\exp(-134.352\tau). \tag{33}$$

The sequence of numbers 8, 9.333, 9.648, ... increases monotonically and tends to the accurate eigenvalue $\pi^2 = 9.8696$. The method of orthogonal projection with respect to the elliptical coordinate $x = \xi$ gives the sequence of numbers 10, 9.8697, 9.8696, ..., which decreases and has the accurate eigenvalue as its lower bound.

The application of numerical methods with respect to all the arguments leads to results in which it is difficult to see exponential stabilization of the solution along the variation in parabolic and even unidirectional elliptical variables [1]. The given hybrid numerical—analytical methods of calculation find temperature fields which are exponentially stabilizing with respect to the time and along the liquid flow. Rapid convergence of the approximate eigenvalues to the accurate value here denotes good agreement of the exponential stabilization in the approximate solutions with the temperature stabilization in the accurate solutions.

CONCLUSIONS

Hybrid numerical—analytical methods of calculating the temperature fields are proposed, on the basis of the accurate mathematical apparatus with respect to unidirectional parabolic variables (over time and along the liquid flow) and approximate numerical or analytic methods of calculation with respect to bidirectional elliptical coordinates (finite-element method, orthogonal projection of the discrepancy, difference method, integral averaging, etc.).

NOTATION

p, s, parameters of double Laplace—Carson transformation in mapping space; t, time; x, y, z, coordinates of current point M; $\langle f \rangle$, integral averaging of function f; w, stabilized velocity profile; $\varphi(X, Fo)$, arbitrary temperature of external medium; $X = (1/Re)(x/R)$; $Pe = w_{me}R/a$, Peclet number; R, tube radius or height of slot channel; $Bi = \alpha R/\lambda$, Biot number.

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